O(2) model

Low-T:

$$H = -J \sum_{\langle ij \rangle} \cos(\theta_i - \theta_j)$$

$$\simeq \frac{J}{2} \int_x (\nabla \theta)^2$$

Exp. significance:

2D superfluids, 2D Solids, O(2) magnets in 2D.

$$\langle \theta(x)\, \theta(x') \rangle \equiv G(x - x')$$

$$= \frac{1}{(2\pi)^2} \int_{k,k'} \langle \theta(k)\, \theta(k') \rangle e^{ikx} \, e^{ik'x'}$$

$$k = -k'$$

$$\Rightarrow = \frac{1}{(2\pi)^2} \int_k \langle |\theta(k)|^2 \rangle e^{ik(x-x')}$$

$$\langle |\theta(k)|^2 \rangle = \frac{\int |\theta(k)|^2 \, e^{-\frac{J}{2} k^2 |\theta(k)|^2} \, d\theta(k)}{\int e^{-\frac{J}{2} k^2 |\theta(k)|^2} \, d\theta(k)}$$

$$= \frac{1}{2 J k^2}$$

$$\Rightarrow \langle \theta(x)\, \theta(x') \rangle = \frac{1}{(2\pi)^2} \frac{1}{J} \int \frac{e^{i\vec{k}\cdot(\vec{x}-\vec{x}')} \, k \, dk \, d\theta}{k^2}$$

$$\Rightarrow G(r) - G(0) = \frac{1}{(2\pi)^2 J} \int_0^{2\pi} \int_0^{a^{-1}} \frac{\left[e^{ikr\cos\theta} - 1 \right] dk \, d\theta}{k}$$

(a = lattice spacing)

On dimensional grounds,

$$G(r) - G(0) \sim -\frac{1}{2\pi J}\int^{r/a}\frac{dk}{k} \sim -\frac{1}{2\pi J}\log\left(\frac{r}{a}\right)$$

Note that the RHS is just the soln to the Green's f^n for Laplace eqn. in 2D.

More relevant and physical is the correlation f^n

$$\left\langle e^{i\theta(\vec{x})}\, e^{-i\theta(\vec{0})}\right\rangle$$

$$= \frac{\int D\theta\; e^{i[\theta(\vec{x}) - \theta(\vec{0})] - \frac{J}{2}\int_x (\nabla\theta)^2\, d^2x}}{\int D\theta\; e^{-\frac{J}{2}\int_x (\nabla\theta)^2}}$$

for any Gaussian prob. dist., [All higher order cumulants vanish]

$$\langle e^{ix}\rangle = e^{-\frac{1}{2}\langle x^2\rangle} \quad \text{(homework)}.$$

$$\Rightarrow \left\langle e^{i[\theta(\vec{x}) - \theta(\vec{0})]}\right\rangle = e^{-\frac{1}{2}[\theta(\vec{x}) - \theta(\vec{0})]^2}$$

$$= e^{[G(x) - G(0)]} = e^{-\frac{1}{2\pi J}\log(|\vec{x}|)}$$

$$\sim \frac{1}{|x|^{1/2\pi J}}$$

homework: calculate $\left\langle e^{i\theta(x) - i\theta(0)}\right\rangle$ 'directly' by performing the path integral.

More generally,

$$e^{i\,n[\theta(x) - \theta(0)]} \sim \frac{1}{|x|^{n^2/2\pi J}}$$

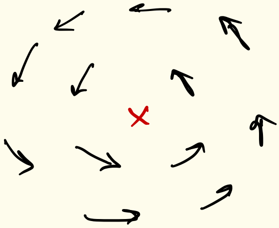
So now we have established that when

$$J \ll 1 : \left\langle e^{i[\theta(x) - \theta(0)]} \right\rangle \sim e^{-|x| \log\left(\frac{1}{J}\right)}$$

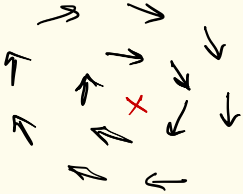
$$J \gg 1 : \left\langle e^{i[\theta(x) - \theta(0)]} \right\rangle \sim \frac{1}{|x|^{1/2\pi J}}$$

Therefore, theremust be a critical J_c below which the behavior changes from exponential to power law. On dimensional grounds $J_c \approx O(1)$. It turns out that vortices play a central role in this transition between the two regimes.

Since the order parameter is a two-comp. vector $(\cos\theta, \sin\theta)$, and we are in two space dim, the system admits topological defects called vortices:



vorticity = +1



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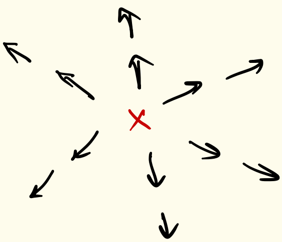
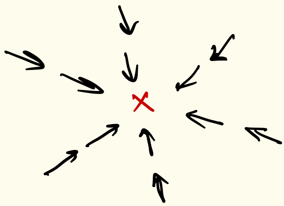


vorticity = +1



vorticity = +1

$$\text{vorticity} = \frac{1}{2\pi} \oint_C \vec{\nabla}\theta \cdot \vec{dl}$$

Contour C encloses the vortex core.

Homework: plot config. with vorticity = −1, +2, −2.

Energy cost of a single vortex:

$$\Delta E = \frac{J}{2}\int_a^L d^2x\,(\nabla\theta)^2 \qquad |\nabla\theta| \sim \frac{n}{|\vec{x}|}$$

where n is the vorticity.

$$= \frac{J}{2} n^2 \int_a^L \frac{d^2x}{x^2}$$

$$\sim J n^2 \log\left(\frac{L}{a}\right)$$

Entropy of a single vortex $\sim \log L$

$$\Delta F = \Delta E - TS$$

$$\sim \log\left(\frac{L}{a}\right)[J-1] \qquad (T \equiv 1)$$

$\Rightarrow$ when $J > 1$, vortices are suppressed.

A more accurate calculation yields $\boxed{J_c = \frac{2}{\pi}}$

J_c $\qquad$ J

$\langle e^{i(\theta(x)-\theta(0))}\rangle \sim e^{-|x|}$ $\qquad$ $e^{i(\theta(x)-\theta(0))} \sim \frac{1}{|x|^{1/2\pi J}}$

At $J = J_c$: $\quad e^{i(\theta(x)-\theta(0))} \sim \frac{1}{|x|^{1/2\pi J_c}} \sim \frac{1}{|x|^{1/4}}$

$\theta(x)$ is not a smooth f^n near the core of a vortex. Infact $n = \frac{\oint \vec{\nabla}\theta \cdot dl}{2\pi}$

$$= \int \frac{[\vec{\nabla} \times \vec{\nabla}\theta]}{2\pi} d^2x$$

$\Rightarrow$ $\frac{\vec{\nabla} \times \vec{\nabla}\theta}{2\pi}$ is the vorticity (i.e. density of vortices). [Note that it is a scalar in 2D]

$\vec{\nabla} \times \vec{\nabla} f$ would vanish for any smooth f^n f.

but $\theta(\vec{x})$ is evidently not smooth near a vortex.

A vortex resembles a point charge.

One can cast the O(2) model as a theory of these charges (= vortices) interacting via a guage field, similar to electromagnetism.

To proceed,

$$Z = \int D\theta \, e^{-\frac{J}{2}\int(\nabla\theta)^2}$$

where crucially we allow θ to be singular (due to vortices)

Let's re-write this as,

$$Z = \int d\theta \, d\vec{\xi} \; e^{-\int \frac{\vec{\xi}^2}{2J} + i \vec{\xi} \cdot \vec{\nabla} \theta}$$

where we have introduced a new vector-field $\vec{\xi}$.

Now write $\theta = \theta_{smooth} + \theta_{vortex}$.

$$\Rightarrow \quad Z = \int d\theta_{smooth} \, d\theta_{vortex} \, d\vec{\xi} \; e^{-\int \vec{\xi}^2/2J + i \vec{\xi} \cdot \vec{\nabla} \theta_{smooth} + i \vec{\xi} \cdot \vec{\nabla} \theta_{vortex}}$$

Integrating out $\theta_{smooth} \Rightarrow \vec{\nabla} \cdot \vec{\xi} = 0$

We can solve this via $\xi_\mu = \varepsilon_{\mu\nu} \partial_\nu a$

where a resembles a guage field.

Performing integration by parts,

$$Z = \int da \; d\rho_{vortex} \; e^{-\int \frac{(\vec{\nabla} a)^2}{2J} + 2\pi i a \rho_{vortex}}$$

where $\rho_{vortex} = \dfrac{\vec{\nabla} \times \vec{\nabla} \theta_{vortex}}{2\pi}$ is the vortex density. This theory resembles electrostatics in 2D. One can integrate out 'a' to obtain

$$Z = \int d\rho_{vortex} \; e^{-J n_{vortex}(x) \log|x - x'| n_{vortex}(x')}$$

$$\left(\sum n_{vortex} = 0 \right)$$

Thus vortices interact logarithmically as expected (recall the energy of a single vortex $\sim J \log(L)$). At low-J, the logarithmic interaction dominates and at J_c, the vortex-anti-vortex pairs unbind.

Physical meaning of gauge field 'a':

a is the only massless field in the above action. Therefore it must be the spin-waves/guassian fluctuations in the low-T phase.

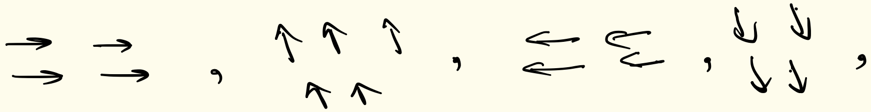
$$\boxed{\text{Fate of } O(2) \text{ model in the presence of anisotropy}}$$

Let's consider perturbing the $O(2)$ model by $\Delta H = h \int_x \cos(n\theta(x))$. This term pins θ to $\frac{2\pi N}{n}$ where $N = 0, 1, \ldots n-1$.

The system no longer has $O(2)$ sym. Instead the sym. is $\mathbb{Z}_n$.

The g.s. will break $\mathbb{Z}_n$ sym. spontaneously.

e.g. when $n = 4$,

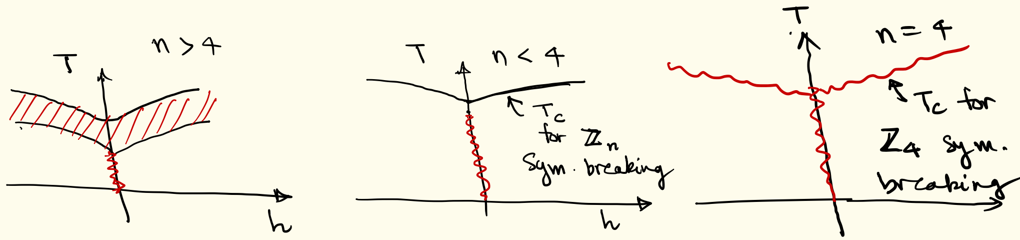


are the four g.s.

What happens to the power-law phase ??

Since discrete sym. breaking is stable to thermal fluct. (recall, e.g. 2D Ising model), at very-low T, one continues to have $\mathbb{Z}_n$ sym. breaking $\Rightarrow \langle e^{i\theta(x)} e^{-i\theta(0)} \rangle \to$ constant.

However, if $n > 4$, one finds an intermediate power-law critical line, similar to the $O(2)$ model.



Above we cheated a little bit. θ is a compact variable i.e. $\theta = \theta + 2\pi$. This is secretly encoded in the fact that ρ_{vortex} must be an integer. This imposes a constraint on the field 'a'.

$$\int d\rho_{vortex}\, e^{2i\pi \rho_{vortex}(x)\, a(x)}$$ restricts

$a(x)$ to be an integer. [Recall the identity $\displaystyle\sum_{n=-\infty}^{+\infty} \delta(x-n) = \sum_{k=-\infty}^{+\infty} e^{2\pi i k x}$].

Let's derive the duality in a bit more rigorous way which will bring out the integer valued-ness of 'a' directly.

Let's start with the actual Hamiltonian

$$H = -J \sum_{x,\mu} \cos(\nabla_\mu \theta_x)$$ where $\nabla_\mu \theta_x \equiv \theta_{x+\mu} - \theta_x$

and $\mu = \hat{x}, \hat{y}$ are the lattice unit vectors.

The partition f^n is

$$Z = \int_0^{2\pi} D\theta \; e^{J\sum_{x,\mu} \cos(\nabla_\mu \theta_x)} .$$

Now we approximate $f(t) = e^{J\cos(t)}$ as

$$\simeq e^J \sum_n e^{-\frac{J}{2}(\theta - 2\pi n)^2} \simeq \int d\ell \; e^J \sum_n e^{-\frac{\ell^2}{2J} + i\ell(\theta - 2\pi n)} \simeq e^J \sum_{\xi=-\infty}^{+\infty} e^{-\frac{\xi^2}{2J} + i\xi t}$$

(This is called Villain approximation)

This approximation maintains the symmetry

$$f(t) = f(t + 2\pi n) .$$

Thus $$Z = \sum_{\xi = -\infty}^{+\infty} \int d\theta \; e^{-\frac{\vec{\xi}^2}{2J} + i\vec{\xi}\cdot\vec{\nabla}\theta}$$ (dropping the pre-factor e^J)

Now we can integrate out θ completely without separating it into smooth and vortex parts. $\Rightarrow \vec{\nabla}\cdot\vec{\xi} = 0$

(This is because θ is effectively non-compact after the Villain approx.)

Solve this in the same manner as before: $\xi_\mu = \varepsilon_{\mu\nu}\partial_\nu a$

a needs to be constrained to integer since ξ is an integer.

$$Z = \int Da \, e^{-\frac{(\vec{\nabla} a)^2}{2J}} \quad \text{with } a \in \mathbb{Z}$$

Let's impose $a \in \mathbb{Z}$ by introducing another field ρ_v.

$$Z = \sum_{\rho_v=-\infty}^{+\infty} \int Da \; e^{-\frac{(\vec{\nabla} a)^2}{2J} + 2\pi i \rho_v a}$$

We identify ρ_v with ρ_{vortex}, thus recovering our earlier result.

Duality in 3D O(2) model

The steps are fairly similar. However, $\vec{\xi}$ is now a 3-component vector. To solve

$$\vec{\nabla} \cdot \vec{\xi} = 0$$

write $\vec{\xi} = \vec{\nabla} \times \vec{a}$ where $\vec{a}$ is again an integer valued field.

$$Z = \int_{-\infty}^{\infty} D\vec{a} \sum_{\vec{j}=-\infty}^{+\infty} e^{-\frac{(\vec{\nabla}\times\vec{a})^2}{2J} + i\vec{j}_v \cdot \vec{a}}$$

where the sum over $\vec{j}_v$ imposes the integer valued ness of $\vec{a}$. One can identify $\vec{j}_v$ with the vortex current. Assuming vortices are bosons, one can write $\vec{j}_v = \text{Im}\left[\phi^* (\vec{\nabla} - i\vec{a})\phi\right]$

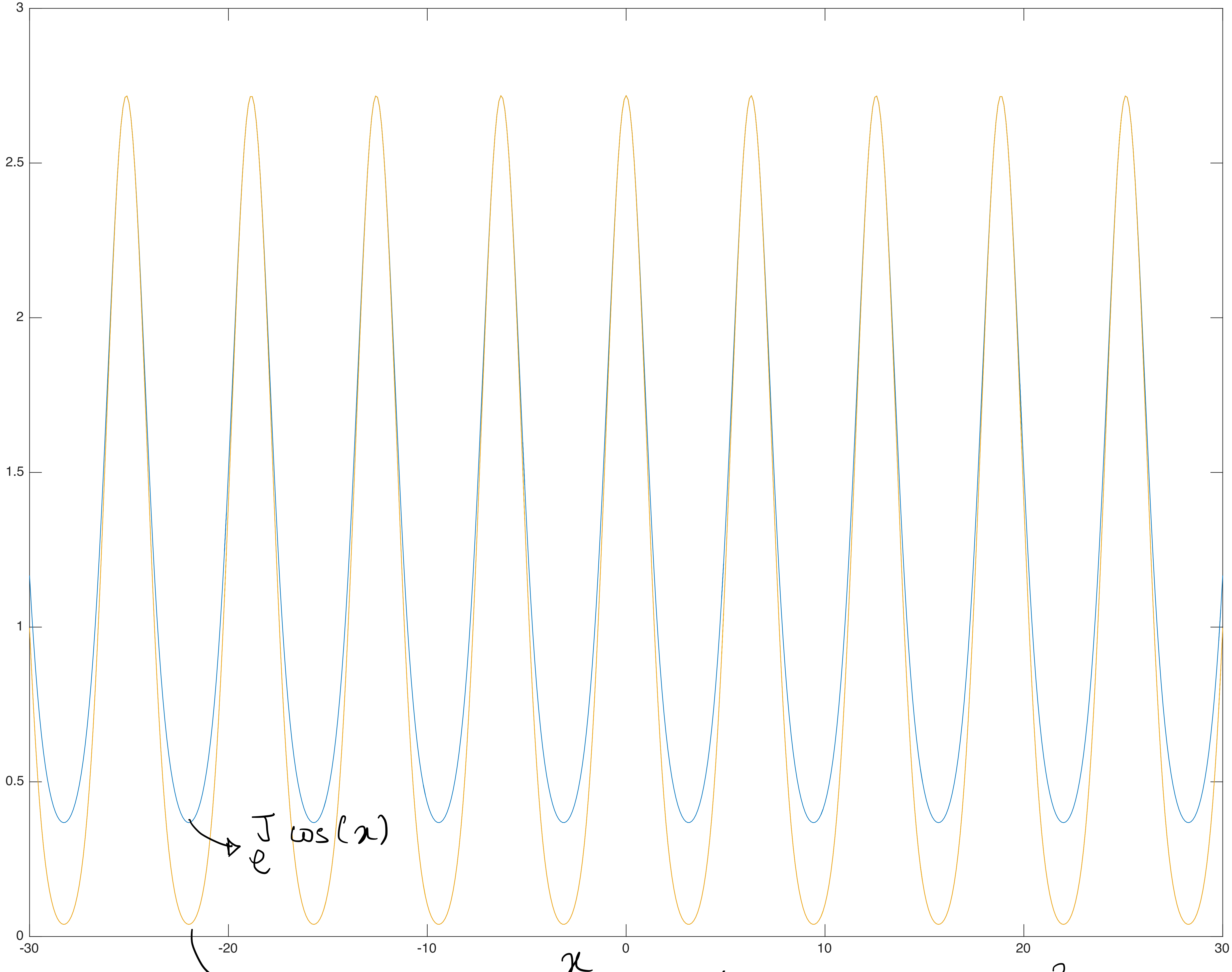
So that the action is,

$$\sim \int |(\vec{\nabla} - i\vec{a})\phi|^2 + (\nabla\times a)^2$$

What happens if one condenses vortices ?

One obtains mass gap for the photon $\vec{a}$. On the other hand when vortices are gapped, the photon is massless. This corresponds to the Goldstone mode of the O(2) spontaneous sym. breaking. Condensing charge two vortices gives a $\mathbb{Z}_2$ guage theory, the subject of our next lecture.

Villain Approx.



Villain approx $\sim \sum_{n=-N}^{+N} e^{-\frac{J}{2}(x-2\pi n)^2}$ with $N = 20$